arXiv:1304.5105v1 [math.PR] 16 Apr 2013

The existence and uniqueness result for Quasilinear Stochastic PDEs with Obstacle under weaker integrability conditions

Laurent DENIS

Université d'Evry-Val-d'Essonne-FRANCE
e-mail: ldenis@univ-evry.fr

Anis MATOUSSI

LUNAM Université, Université du Maine - FRANCE
e-mail: anis.matoussi@univ-lemans.fr

Jing ZHANG

Université d'Evry-Val-d'Essonne -FRANCE
e-mail: jzhang@univ-evry.fr

Abstract: We prove an existence and uniqueness result for quasilinear Stochastic PDEs with Obstacle (in short OSPDE) under a weaker integrability condition on the coefficient and the barrier.

AMS 2000 subject classifications: Primary 60H15; 35R60; 31B150.
Keywords and phrases: Stochastic PDE's, obstacle problems, Itô's formula, comparison theorem, parabolic potential, regular measures.

1. Introduction

In this paper, we consider an obstacle problem for the following parabolic Stochastic PDE (SPDE in short)

$$\begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right) dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j + \nu(t,dx), \\ u_t \geq S_t\,, \\ u_0 = \xi\,. \end{cases} \tag{1}$$

Here a is a matrix defining a symmetric operator on an open domain $\mathcal{O}$, with null boundary condition, f, g, h are random coefficients and S is the given obstacle which is dominated by a given solution, S', of a quasi-linear parabolic SPDE without obstacle.

*The work of the first and third author is supported by the chair *risque de crédit*, Fédération bancaire Française

†The research of the second author was partially supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, the Chair *Derivatives of the Future* sponsored by the Fédération Bancaire Française, and the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon

In a recent work [9] we have proved existence and uniqueness of the solution to this equation (1) under standard Lipschitz hypotheses and L^2-type integrability conditions on the coefficients. Let us recall that the solution is a couple (u,ν), where u is a process with values in the first order Sobolev space and ν is a random regular measure forcing u to stay above S and satisfying a minimal Skohorod condition.
The study of the L^p-norms w.r.t. the randomness of the space-time uniform norm on the trajectories of a stochastic PDE was started by N. V. Krylov in [13], for a more complete overview of existing works on this subject see [7, 8] and the references therein. Concerning the obstacle problem, there are two approaches, a probabilistic one (see [15, 12]) based on the Feynmann-Kac's formula via the backward doubly stochastic differential equations and the analytical one (see [10, 17, 22]) based on the Green function.

In order to give a rigorous meaning to the notion of solution to this obstacle problem and inspired by the works of M. Pierre in the deterministic case (see [18, 19]), we introduce the notion of parabolic capacity. The key point is that in [9], we construct a solution which admits a quasi continuous version hence defined outside a polar set and that regular measures which in general are not absolutely continuous w.r.t. the Lebesgue measure, do not charge polar sets. Moreover, in this reference, we have established an Itô formula which is an essential tool in the present paper in order to get the comparison theorem.

The aim of this paper is to relax the integrability assumptions both on f and S'. More precisely, we prove existence and uniqueness if the random coefficient $f^0(t,x) := f(t,x,0,0)$ belongs to a class $L^{p,q}(\mathcal{O}\times[0,T])$ of functions L^p-integrable in space and L^q-integrable in time for (p,q) such that $(1/p,1/q)$ belongs to a certain interval with end points $p=2, q=1$ and $p=\frac{2^*}{2^*-1}, q=2$ respectively. Here, $2^*>2$ denotes a positive constant which depends on the dimension of the space.

The paper is organized as follows: in section 2 we introduce notations, hypotheses and the basic definitions related to the parabolic potential theory. In section 3, we prove an existence and uniqueness result for the obstacle problem (1) with null Dirichlet condition under a weaker integrability hypothesis on f and S' and also give an estimate of the positive part of the solution. Thanks to this estimate, we establish a comparison theorem for the solutions in section 4. The last section is an Appendix in which we give the proofs of several lemmas.

2. Preliminaries

2.1. $L^{p,q}-$space

Let $\mathcal{O}\subset\mathbb{R}^d$ be an open domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, it is an Hilbert space equipped with the usual scalar product and norm as follows

$$(u,v)=\int_{\mathcal{O}}u(x)v(x)dx, \qquad \| u \|= (\int_{\mathcal{O}}u^2(x)dx)^{1/2}.$$

In general, we shall extend the notation

$$(u,v)=\int_{\mathcal{O}}u(x)v(x)dx,$$

where $u,\ v$ are measurable functions defined on $\mathcal{O}$ such that $uv\in L^1(\mathcal{O})$.
The first order Sobolev space of functions vanishing at the boundary will be denoted by $H_0^1(\mathcal{O})$, its natural scalar product and norm are

$$(u,v)_{H_0^1(\mathcal{O})}=(u,v)+\int_{\mathcal{O}}\sum_{i=1}^{d}\left(\partial_i u\left(x\right)\right)\left(\partial_i v\left(x\right)\right)dx,\ \|u\|_{H_0^1(\mathcal{O})}=\left(\|u\|_2^2+\|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

As usual we shall denote $H^{-1}(\mathcal{O})$ its dual space.
For each $t>0$ and for all real numbers $p,\,q\geq 1$, we denote by $L^{p,q}([0,t]\times\mathcal{O})$ the space of (classes of) measurable functions $u:[0,t]\times\mathcal{O}\longrightarrow\mathbb{R}$ such that

$$\|u\|_{p,q;\,t}:=\left(\int_0^t\left(\int_{\mathcal{O}}|u(s,x)|^p\,dx\right)^{q/p}ds\right)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the use of the essential sup norm.
Now we introduce some other spaces of functions and discuss a certain duality between them. Like in [5] and [7], for self-containeness, we recall the following definitions:
Let $(p_1,q_1),\,(p_2,q_2)\in[1,\infty]^2$ be fixed and set

$$I=I\left(p_1,q_1,p_2,q_2\right):=\Big\{(p,q)\in[1,\infty]^2\,/\,\exists\,\rho\in[0,1]\,s.t.$$

$$\frac{1}{p}=\rho\frac{1}{p_1}+(1-\rho)\frac{1}{p_2},\frac{1}{q}=\rho\frac{1}{q_1}+(1-\rho)\frac{1}{q_2}\Big\}.$$

This means that the set of inverse pairs $\left(\frac{1}{p},\frac{1}{q}\right)$, (p,q) belonging to I, is a segment contained in the square $[0,1]^2$, with the extremities $\left(\frac{1}{p_1},\frac{1}{q_1}\right)$ and $\left(\frac{1}{p_2},\frac{1}{q_2}\right)$.
We introduce:

$$L_{I;t}=\bigcap_{(p,q)\in I}L^{p,q}\left([0,t]\times\mathcal{O}\right).$$

We know that this space coincides with the intersection of the extreme spaces,

$$L_{I;t}=L^{p_1,q_1}\left([0,t]\times\mathcal{O}\right)\cap L^{p_2,q_2}\left([0,t]\times\mathcal{O}\right)$$

and that it is a Banach space with the following norm

$$\|u\|_{I;t}:=\|u\|_{p_1,q_1;t}\vee\|u\|_{p_2,q_2;t}.$$

The other space of interest is the algebraic sum

$$L^{I;t}:=\sum_{(p,q)\in I}L^{p,q}\left([0,t]\times\mathcal{O}\right),$$

which represents the vector space generated by the same family of spaces. This is a normed vector space with the norm

$$\|u\|^{I;t} := \inf\left\{\sum_{i=1}^{n}\|u_i\|_{p_i,q_i;\,t} \ / \ u=\sum_{i=1}^{n}u_i, u_i\in L^{p_i,q_i}\left([0,t]\times\mathcal{O}\right), \, (p_i,q_i)\in I, \, i=1,...n; \, n\in\mathbb{N}^*\right\}.$$

Clearly one has $L^{I;t}\subset L^{1,1}\left([0,t]\times\mathcal{O}\right)$ and $\|u\|_{1,1;t}\leq c\,\|u\|^{I;t}$, for each $u\in L^{I;t}$, with a certain constant $c>0$.
We also remark that if $(p,q)\in I$, then the conjugate pair (p',q'), with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$, belongs to another set, I', of the same type. This set may be described by

$$I'=I'\left(p_1,q_1,p_2,q_2\right):=\left\{\left(p',q'\right) / \ \exists\,(p,q)\in I \ s.t. \ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1\right\}$$

and it is not difficult to check that $I'\left(p_1,q_1,p_2,q_2\right)=I\left(p_1',q_1',p_2',q_2'\right)$, where p_1',q_1',p_2' and q_2' are defined by $\frac{1}{p_1}+\frac{1}{p_1'}=\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{p_2}+\frac{1}{p_2'}=\frac{1}{q_2}+\frac{1}{q_2'}=1$.
Moreover, by Hölder's inequality, it follows that one has

$$\int_0^t\int_{\mathcal{O}}u\left(s,x\right)v\left(s,x\right)dxds\leq\|u\|_{I;t}\,\|v\|^{I';t}\,, \tag{2}$$

for any $u\in L_{I;t}$ and $v\in L^{I';t}$. This inequality shows that the scalar product of $L^2\left([0,t]\times\mathcal{O}\right)$ extends to a duality relation for the spaces $L_{I;t}$ and $L^{I';t}$.
Now let us recall that the Sobolev inequality states that

$$\|u\|_{2^*}\leq c_S\,\|\nabla u\|_2\,, \tag{3}$$

for each $u\in H_0^1\left(\mathcal{O}\right)$, where $c_S>0$ is a constant that depends on the dimension and $2^*=\frac{2d}{d-2}$ if $d>2$, while 2^* may be any number in $]2,\infty[$ if $d=2$ and $2^*=\infty$ if $d=1$. Therefore one has

$$\|u\|_{2^*,2;t}\leq c_S\,\|\nabla u\|_{2,2;t}\,,$$

for each $t\geq 0$ and each $u\in L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathcal{O}\right)\right)$. If $u\in L^\infty_{loc}\left(\mathbb{R}_+;L^2\left(\mathcal{O}\right)\right)\bigcap L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathcal{O}\right)\right)$, one has

$$\|u\|_{2,\infty;t}\vee\|u\|_{2^*,2;t}\leq c_1\left(\|u\|^2_{2,\infty;t}+\|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}},$$

with $c_1=c_S\vee 1$.
One particular case of interest for us in relation with this inequality is when $p_1=2,q_1=\infty$ and $p_2=2^*,q_2=2$. If $I=I\left(2,\infty,2^*,2\right)$, then the corresponding set of associated conjugate numbers is $I'=I'\left(2,\infty,2^*,2\right)=I\left(2,1,\frac{2^*}{2^*-1},2\right)$, where for $d=1$ we make the convention that $\frac{2^*}{2^*-1}=1$. In this particular case we shall use the notation $L_{\#;t}:=L_{I;t}$ and $L^*_{\#;t}:=L^{I';t}$ and the respective norms will be denoted by

$$\|u\|_{\#;t}:=\|u\|_{I;t}=\|u\|_{2,\infty;t}\vee\|u\|_{2^*,2;t}\,,\ \|u\|^*_{\#;t}:=\|u\|^{I';t}.$$

Thus we may write

$$\|u\|_{\#;t}\leq c_1\left(\|u\|^2_{2,\infty;t}+\|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}}, \tag{4}$$

for any $u \in L^{\infty}_{loc}\left(\mathbb{R}_+; L^2\left(\mathcal{O}\right)\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H^1_0\left(\mathcal{O}\right)\right)$ and $t \geq 0$ and the duality inequality becomes

$$\int_0^t \int_{\mathcal{O}} u\left(s,x\right) v\left(s,x\right) dxds \leq \|u\|_{\#;t} \|v\|^*_{\#;t}\,, \tag{5}$$

for any $u \in L_{\#;t}$ and $v \in L^*_{\#;t}$.

2.2. Hypotheses

We consider a sequence $((B^i(t))_{t\geq 0})_{i\in\mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions.
Let A be a symmetric second order differential operator defined on the open subset $\mathcal{O} \subset \mathbb{R}^d$, with domain $\mathcal{D}(A)$, given by

$$A := -L = -\sum_{i,j=1}^{d} \partial_i(a^{i,j}\partial_j).$$

We assume that $a = (a^{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a^{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2,\ \forall x \in \mathcal{O},\ \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants. The energy associated with the matrix a will be denoted by

$$\mathcal{E}\left(w,v\right) = \sum_{i,j=1}^{d} \int_{\mathcal{O}} a^{i,j}(x)\partial_i w(x)\partial_j v(x)\, dx, \quad \forall w,\, v \in H^1_0(\mathcal{O}). \tag{6}$$

We consider the quasilinear stochastic partial differential equation (1) with initial condition $u(0,\cdot) = \xi(\cdot)$ and Dirichlet boundary condition $u(t,x) = 0,\ \forall\ (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}$.
We assume that we have predictable random functions

$$\begin{aligned}
&f : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},\\
&g = (g_1, ..., g_d) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d,\\
&h = (h_1, ..., h_i, ...) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*}.
\end{aligned}$$

We define

$f(\cdot,\cdot,\cdot,0,0) := f^0,\ \ g(\cdot,\cdot,\cdot,0,0) := g^0 = (g^0_1, ..., g^0_d)$ and $h(\cdot,\cdot,\cdot,0,0) := h^0 = (h^0_1, ..., h^0_i, ...)$.

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,\omega,x,y,z)|^2 = \sum_{i=1}^{+\infty} |h_i(t,\omega,x,y,z)|^2.$$

Remark 2.1. *Let us note that this general setting of the SPDE* (1) *we consider, encompasses the case of an SPDE driven by a space-time noise, colored in space and white in time as in [21] for example (see also Example 1 in [9]).*

Assumption (H): There exist non-negative constants $C,\ \alpha,\ \beta$ such that for almost all ω, the following inequalities hold for all $(x,y,z,t) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$:

1. $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \le C(|y-y'| + |z-z'|)$,
2. $|g(t,\omega,x,y,z) - g(t,\omega,x,y',z')| \le C|y-y'| + \alpha|z-z'|$,
3. $|h(t,\omega,x,y,z) - h(t,\omega,x,y',z')| \le C|y-y'| + \beta|z-z'|$,
4. the contraction property: $2\alpha + \beta^2 < 2\lambda$.

Moreover we introduce some integrability conditions on the coefficients $f^0,\ g^0,\ h^0$ and the initial data ξ : we fix a terminal time $T>0$.

Assumption (HI2)

$$E\left(\|\xi\|_2^2 + \left\|f^0\right\|_{2,2;t}^2 + \left\||g^0|\right\|_{2,2;t}^2 + \left\||h^0|\right\|_{2,2;t}^2\right) < \infty,$$

for each $t \in [0,T]$.

Assumption (HI#)

$$E\left(\|\xi\|_2^2 + \left(\left\|f^0\right\|_{\#;t}^*\right)^2 + \left\||g^0|\right\|_{2,2;t}^2 + \left\||h^0|\right\|_{2,2;t}^2\right) < \infty,$$

for each $t \in [0,T]$.

Remark 2.2. *Note that* $(2,1)$ *is the pair of conjugates of the pair* $(2,\infty)$ *and so* $(2,1)$ *belongs to the set* I' *which defines the space* $L_{\#;t}^*$. *Since* $\|v\|_{2,1;t} \le \sqrt{t}\,\|v\|_{2,2;t}$ *for each* $v \in L^{2,2}([0,t]\times\mathcal{O})$, *it follows that*

$$L^{2,2}([0,t]\times\mathcal{O}) \subset L^{2,1;t} \subset L_{\#;t}^*,$$

and $\|v\|_{\#;t}^* \le \sqrt{t}\,\|v\|_{2,2;t}$, *for each* $v \in L^{2,2}([0,t]\times\mathcal{O})$. *This shows that the condition **(HI#)** is weaker than **(HI2)**.*

2.3. Weak solutions

We now introduce $\mathcal{H}_T$, the space of $H_0^1(\mathcal{O})$-valued predictable processes $(u_t)_{t\in[0,T]}$ such that

$$\left(E \sup_{0\le s\le T} \|u_s\|_2^2 + \int_0^T E\,\mathcal{E}(u_s)\,ds\right)^{1/2} < \infty\,.$$

The space of test functions is the algebraic tensor product $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ denotes the space of all real infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Now we recall the definition of the regular measure which has been defined in [9].
$\mathcal{K}$ denotes $L^\infty([0,T];L^2(\mathcal{O}))\cap L^2([0,T];H^1_0(\mathcal{O}))$ equipped with the norm:

$$\begin{aligned}\| v \|^2_{\mathcal{K}} &= \| v \|^2_{L^\infty([0,T];L^2(\mathcal{O}))} + \| v \|^2_{L^2([0,T];H^1_0(\mathcal{O}))}\\ &= \sup_{t\in[0,T[} \| v_t \|^2 + \int_0^T \big(\| v_t \|^2 + \mathcal{E}(v_t)\big)\, dt.\end{aligned}$$

$\mathcal{C}$ denotes the space of continuous functions with compact support in $[0,T[\times\mathcal{O}$ and finally:

$$\mathcal{W} = \{\varphi \in L^2([0,T];H^1_0(\mathcal{O}));\ \frac{\partial\varphi}{\partial t} \in L^2([0,T];H^{-1}(\mathcal{O}))\},$$

endowed with the norm$\| \varphi \|^2_{\mathcal{W}} = \| \varphi \|^2_{L^2([0,T];H^1_0(\mathcal{O}))} + \| \frac{\partial\varphi}{\partial t} \|^2_{L^2([0,T];H^{-1}(\mathcal{O}))}$.
It is known (see [14]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T = \{\varphi\in\mathcal{W};\varphi(T)=0\}$, $\mathcal{W}^+ = \{\varphi\in\mathcal{W};\varphi\geq 0\}$, $\mathcal{W}^+_T = \mathcal{W}_T\cap\mathcal{W}^+$.

Definition 2.1. *An element $v\in\mathcal{K}$ is said to be a* **parabolic potential** *if it satisfies:*

$$\forall\varphi\in\mathcal{W}^+_T,\ \int_0^T -(\frac{\partial\varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt \geq 0.$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

Proposition 2.2. *(Proposition 1.1 in [19]) Let $v\in\mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\mathcal{O}$, denoted by ν^v, such that:*

$$\forall\varphi\in\mathcal{W}_T\cap\mathcal{C},\ \int_0^T (-\frac{\partial\varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt = \int_0^T\int_{\mathcal{O}} \varphi(t,x)d\nu^v.$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\hat{v}$ (resp. $\bar{v}$) : $[0,T]\mapsto L^2(\mathcal{O})$.
Such a Radon measure, ν^v is called **a regular measure** *and we write:*

$$\nu^v = \frac{\partial v}{\partial t} + Av.$$

Definition 2.3. *Let $K\subset[0,T[\times\mathcal{O}$ be compact, $v\in\mathcal{P}$ is said to be $\nu-$superior than 1 on K, if there exists a sequence $v_n\in\mathcal{P}$ with $v_n\geq 1$ a.e. on a neighborhood of K converging to v in $L^2([0,T];H^1_0(\mathcal{O}))$.*

We denote:

$$\mathscr{S}_K = \{v\in\mathcal{P};\ v \textit{ is } \nu-\textit{superior to } 1 \textit{ on } K\}.$$

Proposition 2.4. *(Proposition 2.1 in [19]) Let $K\subset[0,T[\times\mathcal{O}$ compact, then $\mathscr{S}_K$ admits a smallest $v_K\in\mathcal{P}$ and the measure ν^v_K whose support is in K satisfies*

$$\int_0^T\int_{\mathcal{O}} d\nu^v_K = \inf_{v\in\mathcal{P}}\{\int_0^T\int_{\mathcal{O}} d\nu^v;\ v\in\mathscr{S}_K\}.$$

Definition 2.5. *(Parabolic Capacity)*

- *Let $K \subset [0,T[\times\mathcal{O}$ be compact, we define $cap(K) = \int_0^T\int_{\mathcal{O}} d\nu_K^v$;*
- *let $O \subset [0,T[\times\mathcal{O}$ be open, we define $cap(O) = \sup\{cap(K);\ K \subset O \text{ compact}\}$;*
- *for any borelian $E \subset [0,T[\times\mathcal{O}$, we define $cap(E) = \inf\{cap(O);\ O \supset E \text{ open}\}$.*

Definition 2.6. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

Definition 2.7. *(Quasi-continuous)*
A function $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\mathcal{O}$ with:

1. *for all n, the restriction of u_n to the complement of O_n is continuous;*
2. *$\lim_{n\to+\infty} cap\ (O_n) = 0$.*

We say that u admits a quasi-continuous version, if there exists $\tilde{u}$ quasi-continuous such that $\tilde{u} = u$ a.e.

The next proposition, whose proof may be found in [18] or [19] shall play an important role in the sequel:

Proposition 2.8. *Let $K \subset \mathcal{O}$ a compact set, then $\forall t \in [0,T[$*

$$cap(\{t\}\times K) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.
As a consequence, if $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere then it defines uniquely a map from $[0,T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0,T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0,T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T^-}$ is also defined without ambiguity.

Remark 2.3. *The previous proposition applies if for example u is quasi-continuous.*

We end this part by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [19]):

Lemma 2.9. *If $v^n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2([0,T];H_0^1(\mathcal{O}))$; if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$. Then*

$$\lim_{n\to+\infty}\int_0^T\int_{\mathcal{O}} u d\nu^{v^n} = \int_0^T\int_{\mathcal{O}} u d\nu^v.$$

We now give the assumptions on the obstacle that we shall need in the different cases that we shall consider.

Assumption (O): The obstacle $S : [0,T]\times\Omega\times\mathcal{O} \to \mathbb{R}$ is an adapted random field almost surely quasi-continuous, in the sense that for P-almost all $\omega \in \Omega$, the map $(t,x) \to S_t(\omega,x)$ is quasi-continuous. Moreover, $S_0 \le \xi$ P-almost surely and S is controlled by the solution of an SPDE, i.e. $\forall t \in [0,T]$,

$$S_t \le S'_t, \quad dP\otimes dt\otimes dx - a.e. \tag{7}$$

where S' is the solution of the linear SPDE

$$\begin{cases} dS'_t &= LS'_t dt + f'_t dt + \sum_{i=1}^d \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB^j_t \\ S'(0) &= S'_0, \end{cases} \tag{8}$$

with null boundary Dirichlet conditions.

Assumption (HO2)

$$E\left(\|\xi\|_2^2 + \left\|f'\right\|_{2,2;T}^2 + \left\||g'|\right\|_{2,2;T}^2 + \left\||h'|\right\|_{2,2;T}^2\right) < \infty.$$

Assumption (HO#)

$$E\left(\|S'_0\|_2^2 + \left(\left\|f'\right\|_{\#;T}^*\right)^2 + \left\||g'|\right\|_{2,2;T}^2 + \left\||h'|\right\|_{2,2;T}^2\right) < \infty.$$

Remark 2.4. *It is well-known that under* **(HO2)** S' *belongs to* $\mathcal{H}_T$*, is unique and satisfies the following estimate:*

$$E \sup_{t\in[0,T]} \| S'_t \|^2 + E\int_0^T \mathcal{E}(S'_t)dt \leq CE\left[\| S'_0 \|^2 + \int_0^T (\| f'_t \|^2 + \| |g'_t| \|^2 + \| |h'_t| \|^2)dt\right], \tag{9}$$

see for example Theorem 8 in [4]. Moreover, as a consequence of Theorem 3 in [9], we know that S' *admits a quasi-continuous version.*
Under the weaker condition **(HO#)**, S' *also exists, is unique and satisfies the following estimate (see Theorem 3 in [7]):*

$$E \sup_{t\in[0,T]} \| S'_t \|^2 + E\int_0^T \mathcal{E}(S'_t)dt \leq CE\left[\| S'_0 \|^2 + \int_0^T \left((\| f'_t \|_{\#}^*)^2 + \| |g'_t| \|^2 + \| |h'_t| \|^2\right) dt\right]. \tag{10}$$

Definition 2.10. *A pair* (u, ν) *is said to be a solution of the problem (1) if*

1. $u \in \mathcal{H}_T$, $u(t,x) \geq S(t,x)$, $dP \otimes dt \otimes dx - a.e.$ *and* $u_0(x) = \xi$, $dP \otimes dx - a.e.$;
2. ν *is a random regular measure defined on* $[0,T[\times\mathcal{O}$;
3. *the following relation holds almost surely, for all* $t \in [0,T]$ *and all* $\varphi \in \mathcal{D}$,

$$\begin{aligned}(u_t, \varphi_t) =& (\xi, \varphi_0) + \int_0^t (u_s, \partial_s \varphi_s) ds - \int_0^t \mathcal{E}(u_s, \varphi_s) ds \\ &- \sum_{i=1}^d \int_0^t (g_s^i(u_s, \nabla u_s), \partial_i \varphi_s) ds + \int_0^t (f_s(u_s, \nabla u_s), \varphi_s) ds \\ &+ \sum_{j=1}^{+\infty} \int_0^t (h_s^j(u_s, \nabla u_s), \varphi_s) dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi_s(x) \nu(dx, ds);\end{aligned} \tag{11}$$

4. u *admits a quasi-continuous version,* $\tilde{u}$*, and we have*

$$\int_0^T \int_{\mathcal{O}} (\tilde{u}(s,x) - S(s,x)) \nu(dx, ds) = 0, \quad P-a.s.$$

Finally, in the sequel, we introduce some constants ϵ, $\delta > 0$, we shall denote by C_ϵ, C_δ some constants depending only on ϵ, δ, typically those appearing in the kind of inequality

$$|ab| \leq \epsilon a^2 + C_\epsilon b^2. \tag{12}$$

3. Main results

In this section, we prove the existence and uniqueness under a weaker integrability on f^0 and S', improving the results obtained in [9], Theorem 4 and then give an Itô formula and estimate for the positive part of the solution, which is a crucial step leading to the comparison theorem. Let us note that these results have been established in the case of SPDE without obstacle (see Section 3 in [7] and [8]).

3.1. Existence and uniqueness

To get the estimates we need, we apply Itô's formula to $u - S'$, in order to take advantage of the fact that $S - S'$ is non-positive and that as u is solution of (1) and S' satisfies (8), $u - S'$ satisfies

$$\begin{cases} d(u_t - S'_t) = \partial_i(a_{i,j}(x)\partial_j(u_t(x) - S'_t(x)))dt + (f(t,x,u_t(x),\nabla u_t(x)) - f'(t,x))dt \\ \ + \partial_i(g_i(t,x,u_t(x),\nabla u_t(x)) - g'_i(t,x))dt + (h_j(t,x,u_t(x),\nabla u_t(x)) - h'_j(t,x))dB^j_t \\ \ + \nu(x,dt), \\ (u-S')_0 = \xi - S'_0\,, \\ u - S' \geq S - S'\,. \end{cases} \tag{13}$$

that is why we introduce the following functions:

$$\begin{aligned} &\bar{f}(t,\omega,x,y,z) = f(t,\omega,x,y+S'_t,z+\nabla S'_t) - f'(t,\omega,x); \\ &\bar{g}(t,\omega,x,y,z) = g(t,\omega,x,y+S'_t,z+\nabla S'_t) - g'(t,\omega,x); \\ &\bar{h}(t,\omega,x,y,z) = h(t,\omega,x,y+S'_t,z+\nabla S'_t) - h'(t,\omega,x). \end{aligned} \tag{14}$$

Let us remark that the Skohorod condition for $u - S'$ is satisfied since

$$\int_0^T\int_{\mathcal{O}}(u_s(x) - S'_s(x)) - (S_s(x) - S'_s(x))\nu(ds,dx) = \int_0^T\int_{\mathcal{O}}(u_s(x) - S_s(x))\nu(ds,dx) = 0.$$

It is obvious that $\bar{f}$, $\bar{g}$ and $\bar{h}$ satisfy the Lipschitz conditions with the same Lipschitz coefficients as f, g and h. Then, using Remark 2.2, we check the integrability conditions for $\bar{f}^0$, $\bar{g}^0$ and $\bar{h}^0$:

$$\begin{aligned} \left\|\bar{f}^0\right\|^*_{\#;T} &= \left\|f(S',\nabla S') - f'\right\|^*_{\#;T} \leq \left\|f(S',\nabla S')\right\|^*_{\#;T} + \left\|f'\right\|^*_{\#;T} \\ &\leq \left\|f^0\right\|^*_{\#;T} + C\left\|S'\right\|^*_{\#;T} + C\left\|\nabla S'\right\|^*_{\#;T} + \left\|f'\right\|^*_{\#;T} \\ &\leq \left\|f^0\right\|^*_{\#;T} + C\sqrt{t}\left\|S'\right\|_{2,2;T} + C\sqrt{T}\left\|\nabla S'\right\|_{2,2;T} + \left\|f'\right\|^*_{\#;T}. \end{aligned}$$

We know that (see Remark 2.4):

$$E\left(\left\|S'\right\|^2_{2,2;T} + \left\|\nabla S'\right\|^2_{2,2;T}\right) < \infty.$$

Hence, for each t, we have

$$E\left(\left\|\bar{f}^0\right\|^*_{\#;T}\right)^2 < \infty.$$

We also have:

$$\begin{aligned}\left\|\bar{g}^0\right\|_{2,2;T} &= \left\|\bar{g}(S',\nabla S')-g'\right\|_{2,2;T} \le \left\|\bar{g}(S',\nabla S')\right\|_{2,2;T}+\left\|g'\right\|_{2,2;T}\\ &\le \left\|g^0\right\|_{2,2;T}+C\left\|S'\right\|_{2,2;T}+\alpha\left\|\nabla S'\right\|_{2,2;T}+\left\|g'\right\|_{2,2;T}<\infty.\end{aligned}$$

And the same thing for $\bar{h}$. Hence,

$$E\left(\left(\left\|\bar{f}\right\|^*_{\#;T}\right)^2+\left\|\bar{g}\right\|^2_{2,2;T}+\left\|\bar{h}\right\|^2_{2,2;T}\right)<\infty. \tag{15}$$

We now state the main Theorem of this subsection:

Theorem 3.1. *Under conditions* **(H)**, **(O)**, **(HI#)** *and* **(HO#)**, *the obstacle problem (1) admits a unique solution (u,ν), where u is in $\mathcal{H}_T$ and ν is a random regular measure. We denote by $\mathcal{R}^{\#}(\xi,f,g,h,S)$ the solution of OSPDE* (1) *when it exists and is unique.*

For the proof of this theorem, we need the following two lemmas whose proofs are given in the appendix. The first lemma concerns Itô's formula for the solution of SPDE (1) without obstacle under **(H)** and **(HI#)**. Let us remark that in [7], the existence and uniqueness result has been established but not Itô's formula.

Lemma 3.2. *Under the assumptions* **(H)** *and* **(HI#)**, *the SPDE (1) without obstacle admits a unique solution $u\in\mathcal{H}_T$. Moreover, it satisfies Itô's formula i.e. if $\varphi:\mathbb{R}\to\mathbb{R}$ is a function of class C^2 such that φ'' is bounded and $\varphi'(0)=0$, then the following relation holds almost surely, for all $t\in[0,T]$,*

$$\begin{aligned}&\int_{\mathcal{O}}\varphi\left(u_t\left(x\right)\right)dx+\int_0^t\mathcal{E}\left(\varphi'\left(u_s\right),u_s\right)ds=\int_{\mathcal{O}}\varphi\left(\xi\left(x\right)\right)dx+\int_0^t\left(\varphi'\left(u_s\right),f_s\left(u_s,\nabla u_s\right)\right)ds\\ &-\int_0^t\sum_{i=1}^d\left(\partial_i\left(\varphi'\left(u_s\right)\right),g_{i,s}(u_s,\nabla u_s)\right)ds+\frac{1}{2}\int_0^t\left(\varphi''\left(u_s\right),\left|h_s(u_s,\nabla u_s)\right|^2\right)ds\\ &+\sum_{j=1}^\infty\int_0^t\left(\varphi'\left(u_s\right),h_{j,s}(u_s,\nabla u_s)\right)dB^j_s.\end{aligned} \tag{16}$$

The following lemma will be helpful in showing that the solution to problem (1) is quasi-continuous.

Lemma 3.3. *The following PDE with random coefficient f^0 and zero Dirichlet boundary condition*

$$\begin{cases}dw_t+Aw_tdt=f^0_tdt\\ w_0=0\end{cases} \tag{17}$$

has a unique solution $w\in\mathcal{H}_T$. Moreover, w admits a quasi-continuous version.

Proof of Theorem 3.1. We split the proof in 2 steps:

Step 1. We prove an existence and uniqueness result for the problem (1) under the stronger conditions **(H)**, **(O)**, **(HI2)** and **(HO#)**. The idea of the proof is the same as the proof of Theorem 4 in [9].

We begin with the linear case i.e. we assume that f, g and h do not depend on $(u, \nabla u)$, this implies that $f = f^0$, $g = g^0$ and $h = h^0$. We consider the following penalized equation:

$$d(u_t^n - S_t') = L(u_t^n - S_t')dt + \bar{f}_t dt + \sum_{i=1}^d \partial_i \bar{g}_t^i dt + \sum_{j=1}^{+\infty} \bar{h}_t^j dB_t^j + n(u_t^n - S_t)^- dt$$

where $\bar{f} = f - f'$, $\bar{g} = g - g'$and $\bar{h} = h - h'$. Applying Itô's formula (16) to $(u^n - S')^2$, we have almost surely for all $t \in [0, T]$:

$$\begin{aligned} \| u_t^n - S_t' \|^2 + 2\int_0^t \mathcal{E}(u_s^n - S_s')ds &= \| \xi - S_0' \|^2 + 2\int_0^t ((u_s^n - S_s'), \bar{f}_s)ds \\ &- 2\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S_s'), \bar{g}_s^i)ds + 2\sum_{j=1}^{+\infty} \int_0^t ((u_s^n - S_s'), \bar{h}_s^j)dB_s^j \\ &+ 2\int_0^t \int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^- ds + \int_0^t \| |\bar{h}_s| \|^2 \, ds. \end{aligned}$$

We remark first that:

$$\begin{aligned} &\int_0^t \int_{\mathcal{O}} (u_s^n - S_s')n(u_s^n - S_s)^- ds = \int_0^t \int_{\mathcal{O}} (u_s^n - S_s + S_s - S_s')n(u_s^n - S_s)^- ds \\ &= -\int_0^t \int_{\mathcal{O}} n((u_s^n - S_s)^-)^2 ds + \int_0^t \int_{\mathcal{O}} (S_s - S_s')n(u_s^n - S_s)^- dxds. \end{aligned}$$

The last term in the right member is non-positive because $S_t \leq S_t'$, thus,

$$\begin{aligned} \| u_t^n - S_t' \|^2 &+ 2\int_0^t \mathcal{E}(u_s^n - S_s')ds + 2\int_0^t n \,\| (u_s^n - S_s)^- \|^2 \, ds \leq \| \xi - S_0' \|^2 \\ &+ 2\int_0^t (u_s^n - S_s', \bar{f}_s)ds - 2\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S_s'), \bar{g}_s^i)ds \\ &+ 2\sum_{j=1}^{+\infty} \int_0^t (u_s^n - S_s', \bar{h}_s^j)dB_s^j + \int_0^t \| |\bar{h}_s| \|^2 \, ds, \quad a.s. \end{aligned}$$

Then, Hölder's duality inequality (5) and the relation (4) lead to the following estimates, for all t in $[0, T]$, for any δ, $\epsilon > 0$,

$$\begin{aligned} 2|\int_0^t (u_s^n - S_s', \bar{f}_s)ds| &\leq \delta \left\| u^n - S' \right\|_{\#;T}^2 + C_\delta \left(\left\| \bar{f} \right\|_{\#;T}^* \right)^2 \\ &\leq C\delta \left(\left\| u^n - S' \right\|_{2,\infty;T}^2 + \left\| \nabla(u^n - S') \right\|_{2,2;T}^2 \right) + C_\delta \left(\left\| \bar{f} \right\|_{\#;T}^* \right)^2, \end{aligned}$$

and

$$2|\sum_{i=1}^d \int_0^t (\partial_i(u_s^n - S_s'), \bar{g}_s^i)ds| \leq \epsilon \left\| \nabla(u^n - S') \right\|_{2,2;T}^2 + C_\epsilon \, \| |\bar{g}| \|_{2,2;T}^2 \, .$$

Moreover, thanks to the Burkholder-Davies-Gundy inequality, we get

$$\begin{aligned}
E\sup_{t\in[0,T]}|\sum_{j=1}^{+\infty}\int_0^t (u_s^n-S'_s,\bar{h}_s^j)dB_s^j| &\leq c_1E[\int_0^T\sum_{j=1}^{+\infty}(u_s^n-S'_s,\bar{h}_s^j)^2ds]^{1/2}\\
&\leq c_1E[\int_0^T\sum_{j=1}^{+\infty}\sup_{s\in[0,T]}\|u_s^n-S'_s\|^2\|\bar{h}_s^j\|^2\,ds]^{1/2}\\
&\leq c_1E[\sup_{s\in[0,T]}\|u_s^n-S'_s\|(\int_0^T\||\bar{h}_s|\|^2\,ds)^{1/2}]\\
&\leq \epsilon E\sup_{s\in[0,T]}\|u_s^n-S'_s\|^2+\frac{c_1}{4\epsilon}E\int_0^T\||\bar{h}_s|\|^2\,ds.
\end{aligned}$$

Then using the strict ellipticity assumption and the inequalities above, we get

$$\begin{aligned}
&(1-2\epsilon-C\delta)E\sup_{t\in[0,T]}\|u_t^n-S'_t\|^2+(2\lambda-\epsilon-C\delta)E\int_0^T\|\nabla(u_s^n-S'_s)\|^2\,ds\\
&\leq C(E\|\xi-S'_0\|^2+E(\|\bar{f}\|^*_{\#;T})^2+E\||\bar{g}|\|^2_{2,2;T}+E\||\bar{h}|\|^2_{2,2;T}).
\end{aligned}$$

We take ϵ and δ small enough such that $(1-2\epsilon-C\delta)>0$ and $(2\lambda-\epsilon-C\delta)>0$,

$$E\sup_{t\in[0,T]}\|u_t^n-S'_t\|^2+E\int_0^T\mathcal{E}(u_t^n-S'_t)dt\leq C.$$

Then, to prove the existence and uniqueness in this case, we can follow line by line the proof based on a weak convergence argument given in [9], Theorem 4. The only difference is that now the estimates depend on $\|\bar{f}^0\|_{\#;t}$ instead of $\|\bar{f}^0\|_{2,2;t}$.

Step 2. Now we turn to the general case, i.e. assume **(H)**, **(O)**, **(HI#)** and **(HO#)**. We consider the following SPDE:

$$dw_t+Aw_tdt=f_t^0dt \tag{18}$$

Thus $u-w$ satisfies the following OSPDE:

$$\begin{aligned}
d(u_t-w_t)+A(u_t-w_t)dt &= F_t(u_t-w_t,\nabla(u_t-w_t))dt+divG_t(u_t-w_t,\nabla(u_t-w_t))dt\\
&+ H_t(u_t-w_t,\nabla(u_t-w_t))dB_t+\nu(x,dt),
\end{aligned}$$

where

$$F_t(x,y,z)=f_t(x,y+w,z+\nabla w)-f_t^0(x)$$

$$G_t(x,y,z)=g_t(x,y+w,z+\nabla w)$$

$$H_t(x,y,z)=h_t(x,y+w,z+\nabla w).$$

We can easily check that F, G and H satisfy the same Lipschitz conditions as f, g and h and also $F^0\in L^2(\Omega\times[0,T]\times\mathcal{O};\mathbb{R})$, $G^0\in L^2(\Omega\times[0,T]\times\mathcal{O};\mathbb{R}^d)$ and $H^0\in L^2(\Omega\times[0,T]\times\mathcal{O};\mathbb{R}^{\mathbb{N}^*})$. Moreover, $u-w\geq S-w$ and $S-w\leq S'-w$ where $S'-w$ satisfies the following SPDE:

$$d(S'_t-w_t)+A(S'_t-w_t)dt=(f'_t-f_t^0)dt+divg'_tdt+h'_tdB_t.$$

It is easy to see that $f'-f$, g' and h' satisfy **(HO#)**. Therefore, from Step 1, we know that $(u-w,\nu)$ uniquely exists.
Combining with the existence and uniqueness of w, we deduce that the solution of the problem (1) uniquely exists under the weaker assumptions **(HI#)** and **(HO#)**.
And the quasi-continuity of u comes from the quasi-continuity of w and $u-w$. □

3.2. Estimates of the positive part of the solution

To get the estimate of the solution of (1), firstly, we establish an Itô formula for (u,ν).

Theorem 3.4. *Let (u,ν) be the solution of OSPDE (1) and $\varphi:\mathbb{R}\to\mathbb{R}$ be a function of class C^2 and assume that φ'' is bounded and $\varphi'(0)=0$. Then the following relation holds a.s. for all $t\in[0,T]$:*

$$\int_{\mathcal{O}}\varphi\left(u_t\left(x\right)\right)dx+\int_0^t\mathcal{E}\left(\varphi'\left(u_s\right),u_s\right)ds=\int_{\mathcal{O}}\varphi\left(\xi\left(x\right)\right)dx+\int_0^t\left(\varphi'\left(u_s\right),f_s\left(u_s,\nabla u_s\right)\right)ds$$
$$-\int_0^t\sum_{i=1}^d\left(\partial_i\left(\varphi'\left(u_s\right)\right),g_{i,s}(u_s,\nabla u_s\right)ds+\frac{1}{2}\int_0^t\left(\varphi''\left(u_s\right),|h_s(u_s,\nabla u_s)|^2\right)ds$$
$$+\sum_{j=1}^{\infty}\int_0^t\left(\varphi'\left(u_s\right),h_{j,s}(u_s,\nabla u_s)\right)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi'(u_s)\nu(dxds).$$

Proof. The idea is that we begin with the stronger case, where **(H)**, **(O)**, **(HI2)** and **(HO#)** hold. In this case we have the Itô's formula, see step 1 of the proof of Theorem 3.1. Then using an approximation argument we can obtain the Itô's formula in the general case. More precisely:
We take the function $f_n(\omega,t,x):=f(\omega,t,x,u,\nabla u)-f^0+f_n^0$, where f_n^0, $n\in\mathbb{N}^*$, is a sequence of bounded functions such that $E\left(\|f^0-f_n^0\|_{\#;t}^*\right)^2\to 0,\ as\ n\to+\infty$. We consider the following equation

$$du_t^n(x)+Au_t^n(x)dt=f_t^n(x)dt+div\breve{g}_t(x)dt+\breve{h}_t(x)dB_t+\nu^n(x,dt)$$

where $\breve{g}(\omega,t,x)=g(\omega,t,x,u,\nabla u)$ and $\breve{h}(\omega,t,x)=h(\omega,t,x,u,\nabla u)$. This is a linear equation in u^n so from Theorem 3.1, we know that (u^n,ν^n) uniquely exists.
Applying Itô's formula for the difference of two solutions to $(u^n-u^m)^2$ (see Theorem 6 in [9]), we have, almost surely, for all $t\in[0,T]$,

$$\begin{aligned}\|u_t^n-u_t^m\|^2+2\int_0^t\mathcal{E}(u_s^n-u_s^m)ds &= 2\int_0^t(u_s^n-u_s^m,f_s^n-f_s^m)ds\\ &+ 2\int_0^t\int_{\mathcal{O}}(u_s^n-u_s^m)(\nu^n-\nu^m)(dxds).\end{aligned}$$

Remarking that

$$\int\int(u_n-u_m)(\nu_n-\nu_m)(dxds)=\int\int(S-u_m)\nu_n(dxds)-\int\int(u_n-S)\nu_m(dxds)\leq 0$$

and for $\delta > 0$, we have

$$2\left|\int_0^t (u_s^n - u_s^m, f_s^n - f_s^m)ds\right| \quad \leq \quad \delta \left\|u^n - u^m\right\|_{\#;t}^2 + C_\delta \left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2.$$

Since $\mathcal{E}(u^n - u^m) \geq \lambda \left\|\nabla(u^n - u^m)\right\|_2^2$, we deduce that, for all $t \in [0,T]$, almost surely,

$$\left\|u_t^n - u_t^m\right\|^2 + 2\lambda \left\|\nabla(u^n - u^m)\right\|_{2,2;t}^2 \leq \delta \left\|u^n - u^m\right\|_{\#;t}^2 + C_\delta \left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 \qquad (19)$$

Taking the supremum and the expectation, we get

$$E\left(\left\|u^n - u^m\right\|_{2,\infty;t}^2 + \left\|\nabla(u^n - u^m)\right\|_{2,2;t}^2\right) \leq \delta E \left\|u^n - u^m\right\|_{\#;t}^2 + C_\delta E\left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2.$$

Dominating the term $E\left\|u^n - u^m\right\|_{\#;t}^2$ by using the estimate (4) and taking δ small enough, we obtain the following estimate:

$$E\left(\left\|u_n - u_m\right\|_{2,\infty;t}^2 + \left\|\nabla(u_n - u_m)\right\|_{2,2;t}^2\right) \leq 2C_\delta E\left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 \to 0, \; when \; n, \, m \to \infty$$

Therefore, (u^n) has a limit u in $\mathcal{H}_T$.
Now we want to find the limit of (ν^n): we denote by v^n the parabolic potential associated to ν^n, and $z^n = u^n - v^n$, so z^n satisfies the following SPDE

$$dz_t^n(x) + Az_t^n(x)dt = f_t^n(x)dt - \sum_{i=1}^d \partial_i \breve{g}_t^i(x)dt + \sum_{j=1}^\infty \breve{h}_t^j(x)\, dB_t^j.$$

Applying Itô's formula to $(z^n - z^m)^2$, doing the same calculus as before, we obtain the following relation:

$$E\left(\left\|z_n - z_m\right\|_{2,\infty;t}^2 + \left\|\nabla(z_n - z_m)\right\|_{2,2;t}^2\right) \leq 2C_\delta E\left(\left\|f^n - f^m\right\|_{\#;t}^*\right)^2 \longrightarrow 0, \quad as \; n, \, m \to \infty.$$

As a consequence:

$$E\left(\left\|v_n - v_m\right\|_{2,\infty;t}^2 + \left\|\nabla(v_n - v_m)\right\|_{2,2;t}^2\right) \longrightarrow 0, \quad as \; n, \, m \to \infty.$$

Therefore, (v^n) has a limit v in $\mathcal{H}_T$. So, by extracting a subsequence, we can assume that (v^n) converges to v in $\mathcal{K}$, $P-$almost-surely. Then, it's clear that $v \in \mathcal{P}$, and we denote by ν the regular random measure associated to the potential v. Moreover, we have $P - a.s.$ $\forall \varphi \in \mathcal{W}_t^+$,

$$\begin{aligned}
\int_0^t \int_{\mathcal{O}} \varphi(x,s)\nu(dxds) &= \lim_{n\to\infty} \int_0^t \int_{\mathcal{O}} \varphi(x,s)\nu^n(dxds) \\
&= \lim_{n\to\infty} \int_0^t -(v_s^n, \frac{\partial \varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s^n, \varphi_s)ds \\
&= \int_0^t -(v_s, \frac{\partial \varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s, \varphi_s)ds.
\end{aligned}$$

Hence, (u^n, ν^n) converges to (u, ν). Moreover, by Theorem 3.1, we know that the solution of problem (1) uniquely exists and we apply Itô's formula for (u^n, ν^n) : $\forall t \in [0, T]$,

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi\left(u_t^n(x)\right) dx + \int_0^t \mathcal{E}\left(\varphi'\left(u_s^n\right), u_s^n\right) ds = \int_{\mathcal{O}} \varphi\left(\xi(x)\right) dx + \int_0^t \left(\varphi'\left(u_s^n\right), f_s^n\right) ds \\
&- \int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'\left(u_s^n\right)\right), \breve{g}_s^i\right) ds + \frac{1}{2}\int_0^t \left(\varphi''\left(u_s^n\right), \left|\breve{h}_s\right|^2\right) ds + \sum_{j=1}^\infty \int_0^t \left(\varphi'\left(u_s^n\right), \breve{h}_s^j\right) dB_s^j \\
&+ \int_0^t \int_{\mathcal{O}} \varphi'(u_s^n) \nu^n(dxds) \qquad a.s.
\end{aligned} \tag{20}$$

Now, we pass to the limit as n tends to $+\infty$. First, by using Lemma 2.9 and Skohorod condition, we have

$$\int_0^t \int_{\mathcal{O}} \varphi'(u_s^n)\nu^n(dxds) = \int_0^t \int_{\mathcal{O}} \varphi'(S_s)\nu^n(dxds) \to \int_0^t \int_{\mathcal{O}} \varphi'(S_s)\nu(dxds) = \int_0^t \int_{\mathcal{O}} \varphi'(u_s)\nu(dxds).$$

Moreover,

$$\begin{aligned}
& \left|\int_0^t \left(\varphi'\left(u_s^n\right), f_s^n\right) ds - \int_0^t \left(\varphi'\left(u_s\right), f_s\right) ds\right| \\
\leq\ & \left|\int_0^t \left(\varphi'\left(u_s^n\right) - \varphi'\left(u_s\right), f_s^n\right) ds\right| + \left|\int_0^t \left(\varphi'\left(u_s\right), f_s^n - f_s\right) ds\right| \\
\leq\ & C \left\|u^n - u\right\|_{\#;t} \left\|f^n\right\|_{\#;t}^* + C\left\|u\right\|_{\#;t} \left\|f^n - f\right\|_{\#;t}^* .
\end{aligned}$$

The relation (4) and the strong convergence of (u^n) yield that $E\left\|u^n - u\right\|_{\#;t} \to 0$, as $n \to \infty$. So, by extracting a subsequence, we can assume that the right member in the previous inequality tends to 0 $P-$almost surely as n tends to $+\infty$. So we have

$$\lim_{n\to+\infty} \int_0^t \left(\varphi'\left(u_s^n\right), f_s^n\right) ds = \int_0^t \left(\varphi'\left(u_s\right), f_s\right) ds.$$

The convergences of the other terms in (20) are easily deduced from the strong convergence of (u^n) to u in $\mathcal{H}_T$ and then we deduce the desired formula. $\square$

This yields the estimate of the $\mathcal{H}_T$-norm of u under **(HI#)**:

Proposition 3.5. *Under the same hypotheses and notations as in the previous theorem, we have:*

$$\begin{aligned}
E\left(\|u\|_{2,\infty;t}^2 + \|\nabla u\|_{2,2;t}^2\right) \leq\ & k(t)E(\left\|\xi - S_0'\right\|_2^2 + \left(\left\|\bar{f}^0\right\|_{\#;t}^*\right)^2 + \left\|\bar{g}^0\right\|_{2,2;t}^2 + \left\|\bar{h}^0\right\|_{2,2;t}^2 \\
+\ & \left\|S_0'\right\|_2^2 + \left(\left\|f'\right\|_{\#;t}^*\right)^2 + \left\|g'\right\|_{2,2;t}^2 + \left\|h'\right\|_{2,2;t}^2)
\end{aligned}$$

for each $t \in [0, T]$, where $k(t)$ is a constant that only depends on the structure constants and t.

Proof. Since $(u-S',\nu)=\mathcal{R}(\xi-S'_0,\bar f,\bar g,\bar h,S-S')$, applying the above Itô formula to $(u-S')^2$,we have, almost surely, for all $t\in[0,T]$:

$$\begin{aligned}&\left\|u_t-S'_t\right\|^2+2\int_0^t\mathcal{E}(u_s-S'_s)ds=\left\|\xi-S'_0\right\|^2+2\int_0^t(u_s-S'_s,\bar f_s(u_s-S'_s,\nabla(u_s-S'_s)))ds\\&-2\int_0^t(\nabla(u_s-S'_s),\bar g_s(u_s-S'_s,\nabla(u_s-S'_s))ds+2\int_0^t(u_s-S'_s,\bar h_s(u_s-S'_s,\nabla(u_s-S'_s))dB_s\\&+\int_0^t\left\|\bar h_s(u_s-S'_s,\nabla(u_s-S'_s))\right\|^2ds+2\int_0^t\int_{\mathcal{O}}(u_s-S'_s)(x)\nu(dxds).\end{aligned}\tag{21}$$

Remarking the following relation

$$\int_0^t\int_{\mathcal{O}}(u_s-S'_s)\nu(dxds)\leq\int_0^t\int_{\mathcal{O}}(u_s-S_s)\nu(dxds)=0.$$

The Lipschitz conditions in $\bar g$ and $\bar h$ and Cauchy-Schwarz's inequality lead the following relations: for δ, $\epsilon>0$, we have

$$\begin{aligned}\int_0^t(\nabla(u_s-S'_s),\bar g_s(u_s-S'_s,\nabla(u_s-S'_s))ds\ &\leq\ (\alpha+\epsilon)\left\|\nabla(u-S')\right\|^2_{2,2;t}\\&\quad+c_\epsilon\left\|u-S'\right\|^2_{2,2;t}+c_\epsilon\left\|\bar g^0\right\|^2_{2,2;t},\end{aligned}$$

and

$$\int_0^t\left\|\bar h_s(u_s-S'_s,\nabla(u_s-S'_s))\right\|^2ds\leq(\beta^2+\epsilon)\left\|\nabla(u-S')\right\|^2_{2,2;t}+c_\epsilon\left\|u-S'\right\|^2_{2,2;t}+c_\epsilon\left\|\bar h^0\right\|^2_{2,2;t}.$$

Moreover, the Lipschitz condition in $\bar f$, the duality relation between elements in $L_{\#;t}$ and $L^*_{\#;t}$ (5) and Young's inequality (12) yield the following relation:

$$\begin{aligned}\int_0^t\left(u_s-S'_s,\bar f_s\left(u_s-S'_s,\nabla(u_s-S'_s)\right)\right)ds\ &\leq\ \epsilon\left\|\nabla(u-S')\right\|^2_{2,2;t}+c_\epsilon\left\|u-S'\right\|^2_{2,2;t}\\&\quad+\delta\left\|u-S'\right\|^2_{\#;t}+c_\delta\left(\left\|\bar f^0\right\|^*_{\#;t}\right)^2,\end{aligned}$$

Since $\mathcal{E}(u-S')\geq\lambda\left\|\nabla(u-S')\right\|_2^2$, we deduce from (21) that for all $t\in[0,T]$, almost surely,

$$\begin{aligned}&\left\|u_t-S'_t\right\|_2^2+2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\epsilon\right)\left\|\nabla(u-S')\right\|^2_{2,2;t}\leq\left\|\xi-S'_0\right\|_2^2+\delta\left\|u-S'\right\|^2_{\#;t}\\&+2c_\delta\left(\left\|\bar f^0\right\|^*_{\#;t}\right)^2+2c_\epsilon\left\|\bar g^0\right\|^2_{2,2;t}+c_\epsilon\left\|\bar h^0\right\|^2_{2,2;t}+5c_\epsilon\left\|u-S'\right\|^2_{2,2;t}+2M_t,\end{aligned}\tag{22}$$

where $M_t:=\sum_{j=1}^\infty\int_0^t\left(u_s-S'_s,\bar h^j_s\left(u_s-S'_s,\nabla(u_s-S'_s)\right)\right)dB^j_s$ represents the martingale part. Further, using a stopping procedure while taking the expectation, the martingale part vanishes, so that we get

$$\begin{aligned}&E\left\|u_t-S'_t\right\|_2^2+2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\epsilon\right)E\left\|\nabla(u-S')\right\|^2_{2,2;t}\leq E\left\|\xi-S'_0\right\|_2^2+\delta E\left\|u-S'\right\|^2_{\#;t}\\&+2c_\delta E\left(\left\|\bar f^0\right\|^*_{\#;t}\right)^2+2c_\epsilon E\left\|\bar g^0\right\|^2_{2,2;t}+c_\epsilon E\left\|\bar h^0\right\|^2_{2,2;t}+5c_\epsilon\int_0^tE\left\|u_s-S'_s\right\|_2^2ds.\end{aligned}$$

Then we choose $\epsilon = \frac{1}{5}\left(\lambda-\alpha-\frac{\beta^2}{2}\right)$, set $\gamma=\lambda-\alpha-\frac{\beta^2}{2}$ and apply Gronwall's lemma obtaining

$$E\left\|u_t-S'_t\right\|_2^2+\gamma E\left\|\nabla(u-S')\right\|_{2,2;t}^2 \leq \left(\delta E\left\|u-S'\right\|_{\#;t}^2+E\left[F\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)\right]\right)e^{5c_\epsilon t} \tag{23}$$

where $F\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)=\left(\|\xi-S'_0\|^2+2c_\delta\left(\left\|\bar{f}^0\right\|_{\#;t}^*\right)^2+2c_\epsilon\left\|\bar{g}^0\right\|_{2,2;t}^2+c_\epsilon\left\|\bar{h}^0\right\|_{2,2;t}^2\right)$. As a consequence one gets

$$E\left\|u-S'\right\|_{2,2;t}^2 \leq \frac{1}{5c_\epsilon}\left(\delta E\left\|u-S'\right\|_{\#;t}^2+E\left[F\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)\right]\right)\left(e^{5c_\epsilon t}-1\right). \tag{24}$$

Now we return to the inequality (22) and take the supremum, getting

$$\left\|u-S'\right\|_{2,\infty;t}^2 \leq \delta\left\|u-S'\right\|_{\#;t}^2+F\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)+5c_\epsilon\left\|u-S'\right\|_{2,2;t}^2+2\sup_{s\leq t}M_s \tag{25}$$

We would like to take the expectation in this relation and for that reason we need to estimate the bracket of the martingale part,

$$\begin{aligned}\langle M\rangle_t^{\frac{1}{2}} &\leq \left\|u-S'\right\|_{2,\infty;t}\left\|\bar{h}(u-S',\nabla(u-S'))\right\|_{2,2;t}\\ &\leq \eta\left\|u-S'\right\|_{2,\infty;t}^2+c_\eta\left(\left\|u-S'\right\|_{2,2;t}^2+\left\|\nabla(u-S')\right\|_{2,2;t}^2+\left\|\bar{h}^0\right\|_{2,2;t}^2\right)\end{aligned}$$

with η another small parameter to be properly chosen. Using this estimate and the inequality of Burkholder-Davis-Gundy we deduce from the inequality (25):

$$\left(1-2C_{BDG}\eta\right)E\left\|u-S'\right\|_{2,\infty;t}^2 \leq \delta E\left\|u-S'\right\|_{\#;t}^2+E\left[F\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)\right]$$

$$+\left(5c_\varepsilon+2C_{BDG}c_\eta\right)E\left\|u-S'\right\|_{2,2;t}^2+2C_{BDG}c_\eta E\left\|\nabla(u-S')\right\|_{2,2;t}^2+2C_{BDG}c_\eta E\left\|\bar{h}^0\right\|_{2,2;t}^2$$

where C_{BDG} is the constant corresponding to the Burkholder-Davis-Gundy inequality. Further we choose the parameter $\eta=\frac{1}{4C_{BDG}}$ and combine this estimate with (23) and (24) to deduce an estimate of the form:

$$\begin{aligned}E\left(\left\|u-S'\right\|_{2,\infty;t}^2+\left\|\nabla(u-S')\right\|_{2,2;t}^2\right) &\leq \delta c_2(t)E\left\|u-S'\right\|_{\#;t}^2\\ &+ c_3(\delta,t)E\left[R\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)\right]\end{aligned}$$

where $R\left(\delta,\xi-S'_0,\bar{f}^0,\bar{g}^0,\bar{h}^0,t\right)=\left(\|\xi-S'_0\|^2+\left(\left\|\bar{f}^0\right\|_{\#;t}^*\right)^2+\left\|\bar{g}^0\right\|_{2,2;t}^2+\left\|\bar{h}^0\right\|_{2,2;t}^2\right)$ and $c_3(\delta,t)$ is a constant that depends on δ and t, while $c_2(t)$ is independent of δ. Dominating the term $E\|u-S'\|_{\#;t}^2$ by using the estimate (4) and then choosing $\delta=\frac{1}{2c_1^2c_2(t)}$, we get the following estimate:

$$E\left(\left\|u-S'\right\|_{2,\infty;t}^2+\left\|\nabla(u-S')\right\|_{2,2;t}^2\right)\leq k(t)E\left(\left\|\xi-S'_0\right\|_2^2+\left(\left\|\bar{f}^0\right\|_{\#;t}^*\right)^2+\left\|\bar{g}^0\right\|_{2,2;t}^2+\left\|\bar{h}^0\right\|_{2,2;t}^2\right).$$

Combining with the estimate for S' (see Remark 2.4), we obtain the estimate asserted by our proposition. □

In the following Proposition, we establish a crucial relation for the positive part of u:

Proposition 3.6. *Under the hypotheses of Theorem 3.4 with same notations, the following relation holds a.s. for all $t \in [0,T]$:*

$$\begin{aligned}\int_{\mathcal{O}}(u_t^+(x))^2dx + 2\int_0^t \mathcal{E}(u_s^+)ds &= \int_{\mathcal{O}}(\xi^+(x))^2dx + 2\int_0^t (u_s^+, f_s(u_s,\nabla u_s))ds \\ &- 2\int_0^t (\nabla u_s^+, g_s(u_s,\nabla u_s))ds + 2\int_0^t (u_s^+, h_s(u_s,\nabla u_s))dB_s \\ &+ \int_0^t \left\|\mathbb{1}_{\{u_s>0\}} h_s(u_s,\nabla u_s)\right\|^2 ds + 2\int_0^t\int_{\mathcal{O}} u_s^+(x)\nu(dxds).\end{aligned}$$

Proof. We approximate $\psi(y) = (y^+)^2$ by a sequence of regular functions: Let φ be an increasing $\mathcal{C}^\infty$ function such that $\varphi(y) = 0$ for any $y \in]-\infty, 1]$ and $\varphi(y) = 1$ for any $y \in [2, \infty[$. We set $\psi_n(y) = y^2\varphi(ny)$, for each $y \in \mathbb{R}$ and all $n \in \mathbb{N}^*$. It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that

$$\lim_{n\to\infty} \psi_n'(y) = 2y^+, \ \lim_{n\to\infty} \psi_n''(y) = 2\cdot \mathbb{1}_{\{y>0\}},$$

for any $y \in \mathbb{R}$. Moreover we have the estimates

$$0 \le \psi_n(y) \le \psi(y), \ 0 \le \psi_n'(y) \le Cy, \ \left|\psi_n''(y)\right| \le C, \tag{26}$$

for any $y \ge 0$ and all $n \in \mathbb{N}^*$, where C is a constant. We have for all $n \in \mathbb{N}^*$ and each $t \in [0,T]$, a.s.,

$$\begin{aligned}&\int_{\mathcal{O}} \psi_n(u_t(x))\,dx + \int_0^t \mathcal{E}\left({\psi_n}'(u_s), u_s\right)ds = \int_{\mathcal{O}} \psi_n(\xi(x))\,dx + \int_0^t \left({\psi_n}'(u_s), f_s(u_s,\nabla u_s)\right)ds \\ &- \int_0^t \sum_{i=1}^d \left({\psi_n}''(u_s)\,\partial_i u_s, g_{i,s}(u_s,\nabla u_s)\right)ds + \frac{1}{2}\int_0^t \left({\psi_n}''(u_s), |h_s(u_s,\nabla u_s)|^2\right)ds \\ &\quad + \sum_{j=1}^\infty \int_0^t \left({\psi_n}'(u_s), h_{j,s}(u_s,\nabla u_s)\right)dB_s^j + \int_0^t\int_{\mathcal{O}} {\psi_n}'(u_s)\,\nu(dxds).\end{aligned} \tag{27}$$

Taking the limit, thanks to the dominated convergence theorem, we know that all the terms except $\int_0^t\int_{\mathcal{O}} {\psi_n}'(u_s)\,\nu(dxds)$ converge. From (26) and (27), it is easy to verify

$$\sup_n \int_0^t\int_{\mathcal{O}} \psi_n'(u_s)\nu(dxds) \le C.$$

Then, by Fatou's lemma, we have

$$\int_0^t\int_{\mathcal{O}} u_s^+(x)\nu(dxds) = \liminf_{n\to\infty}\int_0^t\int_{\mathcal{O}} \psi_n'(u_s)\nu(dxds) < +\infty, \qquad a.s.$$

Hence, the convergence of the last term comes from the dominated convergence theorem. $\square$

Now we prove an estimate for the positive part u^+ of the solution. For this we need the following notations:

$$\begin{aligned}
&\bar{f}^{u-S',0}=\mathbb{1}_{\{u>S'\}}\bar{f}^0,\ \bar{g}^{u-S',0}=\mathbb{1}_{\{u>S'\}}\bar{g}^0,\ \bar{h}^{u-S',0}=\mathbb{1}_{\{u>S'\}}\bar{h}^0,\\
&\bar{f}^{u-S'}=\bar{f}-\bar{f}^0+\bar{f}^{u-S',0},\ \bar{g}^{u-S'}=\bar{g}-\bar{g}^0+\bar{g}^{u-S',0},\ \bar{h}^{u-S'}=\bar{h}-\bar{h}^0+\bar{h}^{u-S',0},\\
&\bar{f}^{u-S',0+}=\mathbb{1}_{\{u>S'\}}\left(\bar{f}^0\vee 0\right),\ (\xi-S'_0)^+=(\xi-S'_0)\vee 0.
\end{aligned} \tag{28}$$

Proposition 3.7. *Under the hypotheses of Proposition 3.6, one has the following estimate:*

$$\begin{aligned}
E\left(\left\|u^+\right\|^2_{2,\infty;t}\right) \;\leq\;& 2k(t)E\Big(\left\|(\xi-S'_0)^+\right\|_2^2+\left(\left\|\bar{f}^{u-S',0+}\right\|^*_{\#;t}\right)^2+\left\|\bar{g}^{u-S',0}\right\|^2_{2,2;t}+\left\|\bar{h}^{u-S',0}\right\|^2_{2,2;t}\\
&+\left\|S'^+_0\right\|^2_2+\left(\left\|f'^{,0+}\right\|^*_{\#;t}\right)^2+\left\|g'^{,0}\right\|^2_{2,2;t}+\left\|h'^{,0}\right\|^2_{2,2;t}\Big)
\end{aligned}$$

for each $t\in[0,T]$, where $k(t)$ is a constant that only depends on the structure constants and t.

Proof. Since $(u-S',\nu)=\mathcal{R}(\xi-S'_0,\bar{f},\bar{g},\bar{h},S-S')$, by Proposition 3.6, we have almost surely $\forall t\in[0,T]$:

$$\begin{aligned}
&\int_{\mathcal{O}}((u_t-S'_t)^+(x))^2dx+2\int_0^t\mathcal{E}((u_s-S'_s)^+)ds\\
&=\int_{\mathcal{O}}((\xi-S'_0)^+(x))^2dx+2\int_0^t((u_s-S'_s)^+,\bar{f}_s(u_s-S'_s,\nabla(u_s-S'_s)))ds\\
&-2\int_0^t(\nabla(u_s-S'_s)^+,\bar{g}_s(u_s-S'_s,\nabla(u_s-S'_s))ds+2\int_0^t((u_s-S'_s)^+,\bar{h}_s(u_s-S'_s,\nabla(u_s-S'_s))dB_s\\
&+\int_0^t\left\|\mathbb{1}_{\{u_s-S'_s>0\}}(\bar{h}_s(u_s-S'_s,\nabla(u_s-S'_s)))\right\|^2ds+2\int_0^t\int_{\mathcal{O}}(u_s-S'_s)^+(x)\nu(dxds).
\end{aligned}$$

As the support of ν is $\{u=S\}$, we have the following relation

$$\int_0^t\int_{\mathcal{O}}(u_s-S'_s)^+\nu(dxds)=\int_0^t\int_{\mathcal{O}}(S_s-S'_s)^+\nu(dxds)=0.$$

Then we repeat word by word the proof of Proposition 3.5, replacing $u-S'$, $\bar{f}$, $\bar{g}$, $\bar{h}$ and $\xi-S'_0$ by $(u-S')^+$, $\bar{f}^{u-S',0+}$, $\bar{g}^{u-S',0}$, $\bar{h}^{u-S',0}$ and $(\xi-S'_0)^+$ respectively. Hence, we get the following estimate:

$$\begin{aligned}
E(\left\|(u-S')^+\right\|^2_{2,\infty;t} \;+\; \left\|\nabla(u-S')^+\right\|^2_{2,2;t})\leq k(t)E\Big(\left\|(\xi-S'_0)^+\right\|_2^2\\
+\left(\left\|\bar{f}^{u-S',0+}\right\|^*_{\#;t}\right)^2+\left\|\bar{g}^{u-S',0}\right\|^2_{2,2;t}+\left\|\bar{h}^{u-S',0}\right\|^2_{2,2;t}\Big).
\end{aligned}$$

Moreover, from Theorem 4 in [7], we know that

$$E\left(\left\|S'^+\right\|^2_{2,\infty;t}+\left\|\nabla S'^+\right\|^2_{2,2;t}\right)\leq k(t)\,E\left(\left\|S'^+_0\right\|^2_2+\left(\left\|f'^{,0+}\right\|^*_{\#;t}\right)^2+\left\|g'^{,0}\right\|^2_{2,2;t}+\left\|h'^{,0}\right\|^2_{2,2;t}\right).$$

where $S_0^{',+} = S_0' \vee 0$, $f^{',0+} = 1\!\!1_{\{S'>0\}}(f' \vee 0)$, $g^{',0} = 1\!\!1_{\{S'>0\}} g'$ and $h^{',0} = 1\!\!1_{\{S'>0\}} h'$. Then with the relation:

$$E\left\|(u)^+\right\|^2_{2,\infty;t} \le 2E\left(\left\|(u-S')^+\right\|^2_{2,\infty;t} + \left\|(S')^+\right\|^2_{2,\infty;t}\right),$$

we get the desired estimate. □

4. Comparison theorem

In this section we will establish a comparison theorem for the solutions of two OSPDE, $(u^1,\nu^1) = \mathcal{R}^\#(\xi^1, f^1, g, h, S^1)$ and $(u^2,\nu^2) = \mathcal{R}^\#(\xi^2, f^2, g, h, S^2)$, where (ξ^i, f^i, g, h, S^i) satisfy assumptions **(H)**, **(O)**, **(HI#)** and **(HO#)**. It is obvious that $(u^1 - S', \nu^1) = \mathcal{R}^\#(\xi^1 - S_0', \bar f^1, \bar g, \bar h, S^1 - S')$ and $(u^2 - S', \nu^2) = \mathcal{R}^\#(\xi^2 - S_0', \bar f^2, \bar g, \bar h, S^2 - S')$, where $\bar f^i$, $\bar g$, $\bar h$ are defined as in (14) and $(\xi^i - S_0', \bar f^i, \bar g, \bar h, S^i - S')$ satisfy assumptions **(H)**, **(O)**, **(HI#)** and **(HO2)**.

Theorem 4.1. *Assume that the following conditions hold*

1. $\xi^1 \le \xi^2,\ dx\otimes dP - a.e.$
2. $f^1(u^1,\nabla u^1) \le f^2(u^1,\nabla u^1),\ dt\otimes dx\otimes dP - a.e.$
3. $S^1 \le S^2,\ dt\otimes dx\otimes dP - a.e.$

Then for almost all $\omega\in\Omega$, $u^1(t,x) \le u^2(t,x)$, q.e.

Proof. We take the functions: $\bar f^{1,n}(\omega,t,x) := \bar f^1(u^1 - S', \nabla(u^1 - S')) - \bar f^{1,0} + \bar f_n^{1,0}$ and $\bar f^{2,n}(\omega,t,x) := \bar f^2(u^2 - S', \nabla(u^2 - S')) - \bar f^{2,0} + \bar f_n^{2,0}$, where $\bar f_n^{1,0}$ and $\bar f_n^{2,0}$, $n\in\mathbb{N}^*$ are two sequences of bounded functions such that $E\big(\|\bar f^{1,0} - \bar f_n^{1,0}\|^*_{\#;t}\big)^2$ and $E\big(\|\bar f^{2,0} - \bar f_n^{2,0}\|^*_{\#;t}\big)^2$ tend to 0 as $n\to\infty$. We consider the following two equations

$$d\bar u_t^{1,n}(x) + A\bar u_t^{1,n}(x)dt = \bar f_t^{1,n}(x)dt + div\breve g_t^1(x)dt + \breve h_t^1(x)dB_t + \nu^{1,n}(x,dt),$$

and

$$d\bar u_t^{2,n}(x) + A\bar u_t^{2,n}(x)dt = \bar f_t^{2,n}(x)dt + div\breve g_t^2(x)dt + \breve h_t^2(x)dB_t + \nu^{2,n}(x,dt),$$

where $\breve g^i(\omega,t,x) = \bar g(\omega,t,x,u^i - S', \nabla(u^i - S'))$ and $\breve h^i(\omega,t,x) = \bar h(\omega,t,x,u^i - S', \nabla(u^i - S'))$ for $i=1,2$. From the proof of Theorem 3.4, we know that $(\bar u^{i,n}, \nu^{i,n})$ converge to $(\bar u^i, \nu^i)$. As $(\bar u^{i,n}, \nu^{i,n}) = \mathcal{R}(\xi^i - S_0', \bar f^{i,n}, \breve g^i, \breve h^i, S^i - S')$ with $(\xi^i - S_0', \bar f^{i,n}, \breve g^i, \breve h^i, S^i - S')$ satisfy assumptions **(H)**, **(O)**, **(HI2)** and **(HO2)**, we have the following Itô's formula for the difference $\hat u^n := \bar u^{1,n} - \bar u^{2,n}$ of the solutions of two OSPDE (see Theorem 6 in [9]), for all $t\in[0,T]$ and any function $\varphi:\mathbb{R}\longrightarrow\mathbb{R}$ of class C^2 with bounded second order derivative such that $\varphi'(0)=0$:

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi(\hat u_t^n(x))dx + \int_0^t \mathcal{E}(\varphi'(\hat u_s^n), \hat u_s^n)ds = \int_{\mathcal{O}} \varphi(\hat\xi(x))dx + \int_0^t (\varphi'(\hat u_s^n), \hat{\bar f}_s^n)ds \qquad (29)\\
&- \sum_{k=1}^d \int_0^t \int_{\mathcal{O}} \varphi''(\hat u_s^n(x))\partial_k(\hat u_s^n(x))\hat{\breve g}_s^k(x)dxds + \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi'(\hat u_s^n(x))\hat{\breve h}_s^j(x)dxdB_s^j\\
&+ \frac12 \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi''(\hat u_s^n(x))(\hat{\breve h}_s^j(x))^2dxds + \int_0^t \int_{\mathcal{O}} \varphi'(\hat u_s^n(x))(\nu^{1,n} - \nu^{2,n})(dx,ds) \quad a.s.
\end{aligned}$$

Then we approximate the function $\psi: \ y \in \mathbb{R} \rightarrow \varphi(y^+)$ by a sequence (ψ_m) of regular functions. Let ζ be a $\mathcal{C}^\infty$ increasing function such that

$$\forall y \in]-\infty, 1], \ \zeta(y) = 0 \ and \ \forall y \in [2, +\infty[, \ \zeta(y) = 1.$$

We set for all n:

$$\forall y \in \mathbb{R}, \quad \psi_m(y) = \varphi(y)\zeta(ny).$$

It is easy to verify that (ψ_m) converges uniformly to the function ψ, (ψ'_m) converges everywhere to the function $(y \rightarrow \varphi'(y^+))$ and (ψ''_m) converges everywhere to the function $(y \rightarrow \mathbb{1}_{\{y>0\}}\varphi''(y^+))$. Moreover we have the estimates:

$$\forall y \in \mathbb{R}^+, \ m \in \mathbb{N}^*, \quad 0 \leq \psi_m(y) \leq \psi(y), \quad 0 \leq \psi'_m(y) \leq Cy, \quad \left|\psi''_m(y)\right| \leq C, \tag{30}$$

where C is a constant. Thanks to (29) we have almost surely, for $t \in [0, T]$,

$$\begin{aligned}
&\int_{\mathcal{O}} \psi_m(\hat{\bar{u}}^n_t(x))dx + \int_0^t \mathcal{E}(\psi'_m(\hat{\bar{u}}^n_s), \hat{\bar{u}}^n_s)ds = \int_{\mathcal{O}} \psi_m(\hat{\xi}(x))dx + \int_0^t (\psi'_m(\hat{\bar{u}}^n_s), \hat{\bar{f}}^n_s)ds \\
&-\sum_{k=1}^d \int_0^t \int_{\mathcal{O}} \psi''_m(\hat{\bar{u}}^n_s(x))\partial_k(\hat{\bar{u}}^n_s(x))\hat{\bar{g}}^k_s(x)dxds + \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \psi'_m(\hat{\bar{u}}^n_s(x))\hat{\bar{h}}^j_s(x)dxdB^j_s \\
&+\frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \psi''_m(\hat{\bar{u}}^n_s(x))(\hat{\bar{h}}^j_s(x))^2 dxds + \int_0^t \int_{\mathcal{O}} \psi'_m(\hat{\bar{u}}^n_s(x))(\nu^{1,n} - \nu^{2,n})(dx, ds)\,.
\end{aligned}$$

Making m tend to $+\infty$ and using the fact that $\mathbb{1}_{\{\hat{\bar{u}}>0\}}\partial_k \hat{\bar{u}}_s = \partial_k \hat{\bar{u}}^+_s$, we get by the dominated convergence theorem the following formula:

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi(\hat{\bar{u}}^+_{n,t}(x))dx + \int_0^t \mathcal{E}(\varphi'(\hat{\bar{u}}^+_{n,s}), \hat{\bar{u}}^+_{n,s})ds = \int_{\mathcal{O}} \varphi(\hat{\xi}^+(x))dx + \int_0^t (\varphi'(\hat{\bar{u}}^+_{n,s}), \hat{\bar{f}}^n_s)ds \\
&-\sum_{k=1}^d \int_0^t \int_{\mathcal{O}} \varphi''(\hat{\bar{u}}^+_{n,s}(x))\partial_k(\hat{\bar{u}}^+_{n,s}(x))\hat{\bar{g}}^k_s(x)dxds + \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi'(\hat{\bar{u}}^+_{n,s}(x))\hat{\bar{h}}^j_s(x)dxdB^j_s \\
&+\frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi''(\hat{\bar{u}}^+_{n,s}(x))\mathbb{1}_{\{\hat{\bar{u}}_{n,s}>0\}}|\hat{\bar{h}}^j_s(x)|^2 dxds + \int_0^t \int_{\mathcal{O}} \varphi'(\hat{\bar{u}}^+_{n,s}(x))\hat{\nu}^n(dx, ds)\,, \ a.s.
\end{aligned}$$

Thanks to the strong convergence of $(\hat{\bar{u}}^n)_n$, by taking n tend to $+\infty$, we obtain:

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi(\hat{\bar{u}}^+_t(x))dx + \int_0^t \mathcal{E}(\varphi'(\hat{\bar{u}}^+_s), \hat{\bar{u}}^+_s)ds = \int_{\mathcal{O}} \varphi(\hat{\xi}^+(x))dx + \int_0^t (\varphi'(\hat{\bar{u}}^+_s), \hat{\bar{f}}_s)ds \\
&-\sum_{k=1}^d \int_0^t \int_{\mathcal{O}} \varphi''(\hat{\bar{u}}^+_s(x))\partial_k(\hat{\bar{u}}^+_s(x))\hat{\bar{g}}^k_s(x)dxds + \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi'(\hat{\bar{u}}^+_s(x))\hat{\bar{h}}^j_s(x)dxdB^j_s \\
&+\frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi''(\hat{\bar{u}}^+_s(x))\mathbb{1}_{\{\hat{\bar{u}}_s>0\}}|\hat{\bar{h}}^j_s(x)|^2 dxds + \int_0^t \int_{\mathcal{O}} \varphi'(\hat{\bar{u}}^+_s(x))\hat{\nu}(dx, ds)\,, \ a.s.
\end{aligned}$$

Taking $\varphi(x)=x^2$, we have almost surely for all $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathcal{O}}(\hat{u}_t^+(x))^2dx+2\int_0^t\mathcal{E}(\hat{u}_s^+)ds=\int_{\mathcal{O}}(\hat{\xi}^+(x))^2dx+2\int_0^t(\hat{u}_s^+,\hat{\bar{f}}_s)ds\\&-2\sum_{k=1}^d\int_0^t\int_{\mathcal{O}}\partial_k\hat{u}_s^+(x)\hat{g}_s^k(x)dxds+2\sum_{j=1}^{+\infty}\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{\bar{h}}_s^j(x)dxdB_s^j\\&+\sum_{j=1}^{+\infty}\int_0^t\int_{\mathcal{O}}\mathbb{I}_{\{\hat{u}_s>0\}}|\hat{\bar{h}}_s^j(x)|^2dxds+2\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{\nu}(dx,ds)\,.\end{aligned}$$

Remarking the following relation

$$\begin{aligned}&\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{\nu}(dxds)=\int_0^t\int_{\mathcal{O}}(u_s^1-S_s'-(u_s^2-S_s'))^+\hat{\nu}(dxds)=\int_0^t\int_{\mathcal{O}}(u^1-u^2)^+\hat{\nu}(dxds)\\&=\int_0^t\int_{\mathcal{O}}(S^1-u^2)^+\nu^1(dxds)-\int_0^t\int_{\mathcal{O}}(u^1-S^2)^+\nu^2(dxds)\leq 0.\end{aligned}$$

Then a similar argument as in the proof of Proposition 3.5 yields the following estimate:

$$E\left(\left\|\hat{u}^+\right\|_{2,\infty;t}^2+\left\|\nabla\hat{u}^+\right\|_{2,2;t}^2\right)\leq k(t)E\left(\left\|\hat{\xi}^+\right\|_2^2+\left(\left\|\hat{\bar{f}}^{\hat{u},0+}\right\|_{\theta;t}^*\right)^2+\left\|\hat{\bar{g}}^{\hat{u},0}\right\|_{2,2;t}^2+\left\|\hat{\bar{h}}^{\hat{u},0}\right\|_{2,2;t}^2\right).$$

This implies $\bar{u}^1\leq\bar{u}^2$ since $\hat{\xi}\leq 0$, $\hat{\bar{f}}^0\leq 0$ and $\hat{\bar{g}}^0=\hat{\bar{h}}^0=0$. Thanks to the uniqueness of the solution of OSPDE (see Theorem 3.1), we know that $\bar{u}^1=u^1-S'$ and $\bar{u}^2=u^2-S'$ which yields the desired result. $\square$

5. Appendix

5.1. Proof of Lemma 3.2

Proof. We take the function $f_n(\omega,t,x):=f(\omega,t,x,u,\nabla u)-f^0+f_n^0$, where $f_n^0,\ n\in\mathbb{N}^*$, is a sequence of bounded functions such that $E\left(\|f^0-f_n^0\|_{\#;t}^*\right)^2\to 0,\ as\ n\to+\infty$. We consider the following equation

$$du_t^n(x)+Au_t^n(x)dt=f_t^n(x)dt+div\breve{g}_t(x)dt+\breve{h}_t(x)dB_t$$

where $\breve{g}(\omega,t,x)=g(\omega,t,x,u,\nabla u)$ and $\breve{h}(\omega,t,x)=h(\omega,t,x,u,\nabla u)$. This is a linear equation in u^n, from [4], we know that u^n uniquely exists.
Applying Itô's formula to $(u^n-u^m)^2$, almost surely, for all $t\in[0,T]$,

$$\|u_t^n-u_t^m\|^2+2\int_0^t\mathcal{E}(u_s^n-u_s^m)ds\ =\ 2\int_0^t(u_s^n-u_s^m,f_s^n-f_s^m)ds.$$

From (5), we have, for $\delta>0$,

$$2\left|\int_0^t(u_s^n-u_s^m,f_s^n-f_s^m)ds\right|\ \leq\ \delta\left\|u^n-u^m\right\|_{\#;t}^2+C_\delta\left(\|f^n-f^m\|_{\#;t}^*\right)^2.$$

Since $\mathcal{E}(u^n-u^m)\geq \lambda \left\|\nabla(u^n-u^m)\right\|_2^2$, we deduce that, for all $t\in[0,T]$, almost surely,

$$\|u_t^n-u_t^m\|^2+2\lambda\left\|\nabla(u^n-u^m)\right\|_{2,2;t}^2\leq \delta\left\|u^n-u^m\right\|_{\#;t}^2+C_\delta\left(\|f^n-f^m\|_{\#;t}^*\right)^2. \quad (31)$$

Taking the supremum and the expectation, we get

$$E\left(\|u^n-u^m\|_{2,\infty;t}^2+\left\|\nabla(u^n-u^m)\right\|_{2,2;t}^2\right)\leq \delta E\left\|u^n-u^m\right\|_{\#;t}^2+C_\delta E\left(\|f^n-f^m\|_{\#;t}^*\right)^2.$$

Dominating the term $E\left\|u^n-u^m\right\|_{\#;t}^2$ by using the estimate (4) and taking δ small enough, we obtain the following estimate:

$$E\left(\|u_n-u_m\|_{2,\infty;t}^2+\left\|\nabla(u_n-u_m)\right\|_{2,2;t}^2\right)\leq C_\delta E\left(\|f^n-f^m\|_{\#;t}^*\right)^2\to 0,\ when\ n,\,m\to\infty.$$

Therefore (u^n) has a limit u in $\mathcal{H}$.
See for example [5], we know that for u^n we have the following Itô's formula, for all $t\in[0,T]$, $P-a.s.$:

$$\begin{aligned}\int_{\mathcal{O}}\varphi\left(u_t^n(x)\right)dx+\int_0^t\mathcal{E}\left(\varphi'\left(u_s^n\right),u_s^n\right)ds=\int_{\mathcal{O}}\varphi\left(\xi(x)\right)dx+\int_0^t\left(\varphi'\left(u_s^n\right),f_s^n\right)ds\\ -\int_0^t\sum_{i=1}^d\left(\partial_i\left(\varphi'\left(u_s^n\right)\right),\breve{g}_s^i\right)ds+\frac12\int_0^t\left(\varphi''\left(u_s^n\right),\left|\breve{h}_s\right|^2\right)ds+\sum_{j=1}^\infty\int_0^t\left(\varphi'\left(u_s^n\right),\breve{h}_s^j\right)dB_s^j.\end{aligned}$$

Now, we pass to the limit as n tend to $+\infty$:

$$\begin{aligned}&\left|\int_0^t\left(\varphi'\left(u_s^n\right),f_s^n\right)ds-\int_0^t\left(\varphi'\left(u_s\right),f_s\right)ds\right|\\ \leq\ &\left|\int_0^t\left(\varphi'\left(u_s^n\right)-\varphi'\left(u_s\right),f_s^n\right)ds\right|+\left|\int_0^t\left(\varphi'\left(u_s\right),f_s^n-f_s\right)ds\right|\\ \leq\ &C\left\|u^n-u\right\|_{\#;t}\left\|f^n\right\|_{\#;t}^*+C\left\|u\right\|_{\#;t}\left\|f^n-f\right\|_{\#;t}^*.\end{aligned}$$

The relation (4) and the strong convergence of (u^n) yield that $E\left\|u^n-u\right\|_{\#;t}\to 0$, as $n\to\infty$. So, by extracting a subsequence, we can assume that the right member in the previous inequality tends to 0 $P-$almost surely as n tends to $+\infty$. So we have

$$\lim_{n\to+\infty}\int_0^t\left(\varphi'\left(u_s^n\right),f_s^n\right)ds=\int_0^t\left(\varphi'\left(u_s\right),f_s\right)ds.$$

The convergence of the other terms are easily deduced from the strong convergence of (u^n) to u in $\mathcal{H}_T$ and yield the desired formula. □

5.2. Proof of Lemma 3.3

Proof. First of all, this equation is a special case of Theorem 3 in [7] hence, we know that w exists is unique and belongs to $\mathcal{H}_T$.
Following M.Pierre [18, 19] and F.Mignot and J.Puel [16], we define

$$\kappa(w,0):=ess\inf\{u\in\mathcal{P};\ u\geq w\ a.e.,\ u(0)\geq 0\}.$$

We consider the following equation:

$$\begin{cases} \dfrac{\partial v_t^n}{\partial t} = Lv_t^n + n(v_t^n - w_t)^- \\ v_0^n = 0 \end{cases} \tag{32}$$

From [16], for almost all $\omega \in \Omega$, we know that $v^n(\omega)$ converges weakly to $v(\omega) := \kappa^\omega(w, 0)$ in $L^2(0, T; H_0^1(\mathcal{O}))$ and that $v(\omega) \geq w(\omega)$.
(17)-(32) yields

$$d(v_t^n - w_t) + A(v_t^n - w_t)dt = (n(v_t^n - w_t)^- - f_t^0)dt$$

so, we have the following relation almost surely, $\forall t \geq 0$,

$$\|v_t^n - w_t\|^2 + 2\int_0^t \mathcal{E}(v_s^n - w_s)ds = 2\int_0^t \int_{\mathcal{O}} (v_s^n - w_s)n(v_s^n - w_s)^- dxds - 2\int_0^t (v_s^n - w_s, f_s^0)ds.$$

The first term is negative and

$$\left| \int_0^t (v_s^n - w_s, f_s^0)ds \right| \leq \delta \|v^n - w\|_{\#;t}^2 + C_\delta \left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

Therefore

$$\|v_t^n - w_t\|_2^2 + 2\lambda \|\nabla(v^n - w)\|_{2,2;t}^2 \leq 2\delta \|v^n - w\|_{\#;t}^2 + 2C_\delta \left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

Taking the supremum and the expectation, we get

$$E\|v^n - w\|_{2,\infty;t}^2 \leq 2\delta E\|v^n - w\|_{\#;t}^2 + 2C_\delta E\left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

Dominating the term $E\|v^n - w\|_{\#;t}^2$ by using the estimate (4) and taking δ small enough, we obtain

$$E\|v^n - w\|_{2,\infty;t}^2 + E\|\nabla(v^n - w)\|_{2,2;t}^2 \leq CE\left(\left\| f^0 \right\|_{\#;t}^* \right)^2.$$

By Fatou's lemma, we have

$$E \sup_{t \in [0,T]} \|\kappa_t - w_t\|^2 + E\int_0^T \mathcal{E}(\kappa_t - w_t)dt \leq CE\int_0^T \left(\left\| f_t^0 \right\|_{\#}^* \right)^2 dt. \tag{33}$$

We now consider a sequence of random functions $(f^{0,n})_{n \in \mathbb{N}^*}$ which belongs in $L^2(\Omega) \otimes C_c^\infty(\mathbb{R}^+) \otimes C_c^\infty(\mathcal{O})$ and such that $E\left\| f^{0,n} - f^0 \right\|_{\#;t}^* \to 0$. Let w^n be the solution of

$$\begin{cases} dw_t^n + Aw_t^n dt = f_t^{0,n}dt \\ w_0^n = 0. \end{cases}$$

Then it's well known that w^n is $P-$almost surely continuous in (t, x) (see for example [1]). Then, we consider a sequence of random open sets

$$\vartheta_n = \{|w^{n+1} - w^n| > \epsilon_n\}, \quad \Theta_p = \bigcup_{n=p}^{+\infty} \vartheta_n,$$

and $\kappa_n = \kappa(\frac{1}{\epsilon_n}(w^{n+1}-w^n),0)+\kappa(-\frac{1}{\epsilon_n}(w^{n+1}-w^n),0)$. From the definition of κ and the relation (see [19]), we get

$$\kappa(|v|) \leq \kappa(v, v^+(0)) + \kappa(-v, v^-(0).)$$

Moreover, κ_n satisfy the conditions of Lemma 3.3 in [19], i.e. $\kappa_n \in \mathcal{P}$ and $\kappa_n \geq 1$ *a.e.* on ϑ_n, therefore, we get the following relation:

$$E[cap\,(\Theta_p)] \leq \sum_{n=p}^{+\infty} E[cap\,(\vartheta_n)] \leq \sum_{n=p}^{+\infty} E\,\|\kappa_n\|_{\mathcal{K}}^2 \leq 2C \sum_{n=p}^{+\infty} \frac{1}{\epsilon_n^2} E \int_0^T \left(\left\| f_t^{0,n+1} - f_t^{0,n} \right\|_{\#}^{*} \right)^2 dt,$$

where the last inequality comes from (33).
By extracting a subsequence, we can consider that

$$E \int_0^T \left(\left\| f_t^{0,n+1} - f_t^{0,n} \right\|_{\#}^{*} \right)^2 dt \leq \frac{1}{2^n}$$

and taking $\epsilon_n = \frac{1}{n^2}$ to get

$$E[cap\ (\Theta_p)] \leq \sum_{n=p}^{+\infty} \frac{2Cn^4}{2^n}.$$

Therefore

$$\lim_{p\to+\infty} E[cap\ (\Theta_p)] = 0.$$

Finally, for almost all $\omega \in \Omega$, $w^n(\omega)$ is continuous in (t,x) on $(\Theta_p(\omega))^c$ and $(w^n(\omega))$ converges uniformly to $w(\omega)$ on $(\Theta_p(\omega))^c$ for all p, hence, $w(\omega)$ is continuous in (t,x) on $(\Theta_p(\omega))^c$, then from the definition of quasi-continuity, we know that $w(\omega)$ admits a quasi-continuous version since $cap(\Theta_p)$ tends to 0 almost surely as p tends to $+\infty$. □

References

[1] Aronson, D.G.: Non-negative solutions of linear parabolic equations. *Annali della Scuola Normale Superiore di Pisa*, Classe di Scienze 3, tome 22 (4), pp. 607-694 (1968).
[2] Aronson D.G. and Serrin J. : Local behavior of solutions of quasi-linear parabolic equations. *Archive for Rational Mechanics and Analysis*, 25, 81-122 (1967).
[3] Bally V., Caballero E., El-Karoui N. and Fernandez, B. : Reflected BSDE's PDE's and Variational Inequalities. *preprint INRIA report* (2004).
[4] Denis L. and Stoïca L.: A general analytical result for non-linear s.p.d.e.'s and applications.Electronic *Journal of Probability*, 9, p. 674-709 (2004).
[5] Denis L., Matoussi A. and Stoïca L.: L^p estimates for the uniform norm of solutions of quasilinear SPDE's. *Probability Theory Related Fields*, 133, 437-463 (2005).
[6] Denis L., Matoussi A. and Stoïca L.: Maximum principle for parabolic SPDE's: first approach. *Stohcastic Partial Differential Equations and Applications VIII, Levico, Jan.* 6-12 (2008).

[7] Denis L., Matoussi A. and Stoïca L.: Maximum Principle and Comparison Theorem for Quasi-linear Stochastic PDE's. *Electronic Journal of Probability*, 14, p. 500-530 (2009).

[8] Denis L. and Matoussi A.: Maximum principle for quasilinear SPDE's on a bounded domain without regularity assumptions. To appear in *Stoch. Proc. and their applications*, (2013).

[9] Denis L., Matoussi A. and Zhang J.: The Obstacle Problem for Quasilinear Stochastic PDEs: Analytical approach. To appear in *Annals of Probability*, (2013).

[10] Donati-Martin C. and Pardoux E.: White noise driven SPDEs with reflection. *Probability Theory and Related Fields*, 95, 1-24 (1993).

[11] El Karoui N., Kapoudjian C., Pardoux E., Peng S., and Quenez M.C.: Reflected Solutions of Backward SDE and Related Obstacle Problems for PDEs. *The Annals of Probability*, 25 (2), 702-737 (1997).

[12] Klimsiak T.: Reflected BSDEs and obstacle problem for semilinear PDEs in divergence form. *Stochastic Processes and their Applications*, 122 (1), 134-169 (2012).

[13] Krylov, N. V. : An analytic approach to SPDEs. *Six Perspectives*, *AMS Mathematical surveys an Monographs*, 64, 185-242 (1999).

[14] Lions J.L. and Magenes E.: Problèmes aux limites non homogènes et applications. *Dunod, Paris* (1968).

[15] Matoussi A. and Stoïca L.: The Obstacle Problem for Quasilinear Stochastic PDE's. *Annals of Probability*, 38, 3, 1143-1179 (2010).

[16] Mignot F. and Puel J.P. : Inéquations d'évolution paraboliques avec convexes dépendant du temps. Applications aux inéquations quasi-variationnelles d'évolution. *Arch. for Rat. Mech. and Ana.*, 64, No.1, 59-91 (1977).

[17] Nualart D. and Pardoux E.: White noise driven quasilinear SPDEs with reflection. *Probability Theory and Related Fields*, 93, 77-89 (1992).

[18] Pierre M.: Problèmes d'Evolution avec Contraintes Unilaterales et Potentiels Parabolique. *Comm. in Partial Differential Equations*, 4(10), 1149-1197 (1979).

[19] Pierre M. : Représentant Précis d'Un Potentiel Parabolique. *Séminaire de Théorie du Potentiel*, Paris, No.5, Lecture Notes in Math. 814, 186-228 (1980).

[20] Revuz, D. and Yor, M. : Continuous Martingales and Brownian Motion. *Springer, third edition*, (1999).

[21] Sanz M. , Vuillermot P. : Equivalence and Hölder Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Ann. I. H. Poincaré PR*, 39 (4), 703-742, (2003).

[22] Xu T.G. and Zhang T.S.: White noise driven SPDEs with reflection: Existence, uniqueness and large deviation principles *Stochatic processes and their applications*, 119, 3453-3470 (2009).

Laurent DENIS
Laboratoire d'Analyse et Probabilités
Université d'Evry Val d'Essonne
23 Boulevard de France
F-91037 Evry Cedex, FRANCE
e-mail: ldenis@univ-evry.fr

Anis MATOUSSI
LUNAM Université, Université du Maine
Fédération de Recherche 2962 du CNRS
Mathématiques des Pays de Loire
Laboratoire Manceau de Mathématiques
Avenue Olivier Messiaen
F-72085 Le Mans Cedex 9, France
email : anis.matoussi@univ-lemans.fr
and
CMAP, Ecole Polytechnique, Palaiseau

Jing ZHANG
Laboratoire d'Analyse et Probabilités
Université d'Evry Val d'Essonne
23 Boulevard de France
F-91037 Evry Cedex, FRANCE
Email: jing.zhang.etu@gmail.com